Lecture 21: Orthogonality cont.

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Theorem Orthogonal sets are linearly independent

but linearly independent sets are not necessarily orthogonal.

PROOF

Let $\{v_1, \dots, v_m\}$ be an orthogonal set.
Look at

$$a_1 \cdot v_1 + \cdots + a_m \cdot v_m = 0$$
$$\Rightarrow v_i \cdot (a_1 \cdot v_1 + \cdots + a_m \cdot v_m) = v_i \cdot 0$$
$$\Rightarrow a_1 \underbrace{v_i \cdot v_1}_{\substack{0 \\ \text{unless } i=1}} + a_2 \underbrace{v_i \cdot v_2}_{\substack{0 \\ \text{unless } i=2}} + \cdots + a_m \underbrace{v_i \cdot v_m}_{\substack{0 \\ \text{unless } i=m}} = v_i = 0$$
$$\Rightarrow a_i \underbrace{v_i \cdot v_i}_{\neq 0} = 0$$

(b/c $v_1 \neq 0$ as a vector in a LI set)

$$\Rightarrow a_i = 0$$

Now, choosing $i = 1, i = 2, \dots, i = m$, we get:
$a_1 = a_2 = \cdots = a_m$

Corollary
An orthogonal spanning set is a basis.

Theorem
Let $\{w_1, \dots, w_m\}$ be an orthogonal basis of a subspace W of $\mathbb{R}^n$.
Then, any $w \in$ W can be written as:

$$w = \ ? \ \cdot w_1 + \ ? \ \cdot w_2 + \cdots + \ ? \ \cdot w_m$$

$$w = \frac{w_1 \cdot w}{w_1 \cdot w_1} \cdot w_1 + \frac{w_2 \cdot w}{w_2 \cdot w_2} \cdot w_2 + \cdots + \frac{w_m \cdot w}{w_m \cdot w_m} \cdot w_m$$

PROOF

Since $\{w_1, \dots w_m\}$ form a basis of W there are $a_1, \dots, a_m \in \mathbb{R}$ such that:

$$w = a_1 w_1 + \cdots + a_m w_m \quad \Big| \ w_i$$

dot product with w_i on both sides

$$\Rightarrow w_i \cdot w = w_i \cdot (a_1 w_1 + \cdots + a_m w_m)$$
$$\Rightarrow w_i \cdot w = a_1 w_i \cdot w_1 + \cdots + a_m w_i \cdot w_m$$
$$\Rightarrow w_i \cdot w = a_i w_i \cdot w_i$$
$$\boxed{\Rightarrow \frac{w_i \cdot w}{w_i \cdot w_i} = a_i}$$

Example:

$$\left\{ \underset{w_1}{\begin{pmatrix}1\\2\\1\end{pmatrix}}, \underset{w_2}{\begin{pmatrix}1\\0\\-1\end{pmatrix}}, \underset{w_3}{\begin{pmatrix}1\\-1\\1\end{pmatrix}} \right\}$$

→ orthogonal basis of $\mathbb{R}^3$

$$\underset{w}{\begin{pmatrix}1\\1\\1\end{pmatrix}} = \ ? \ \cdot \underset{w_1}{\begin{pmatrix}1\\2\\1\end{pmatrix}} + \ ? \ \cdot \underset{w_2}{\begin{pmatrix}1\\0\\-1\end{pmatrix}} + \ ? \ \cdot \underset{w_3}{\begin{pmatrix}1\\-1\\1\end{pmatrix}}$$

$$\frac{w_1 \cdot w}{w_1 \cdot w_1} = \frac{4}{6} = \frac{2}{3}$$
$$\frac{w_2 \cdot w}{w_2 \cdot w_2} = 0$$
$$\frac{w_3 \cdot w}{w_3 \cdot w_3} = \frac{1}{3}$$

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$$\overset{w}{\begin{pmatrix}1\\1\\1\end{pmatrix}} = \frac{2}{3}\cdot\overset{w_1}{\begin{pmatrix}1\\2\\1\end{pmatrix}} + 0\cdot\overset{w_2}{\begin{pmatrix}1\\0\\-1\end{pmatrix}} + \frac{1}{3}\cdot\overset{w_3}{\begin{pmatrix}1\\-1\\1\end{pmatrix}}$$

$$\frac{w_3\cdot w}{w_3\cdot w_3} = \frac{1}{3}$$

19.3 Orthogonal Projections

Definition

Let W be a subspace of $\mathbb{R}^n$ with orthogonal basis $\{w_1,\dots,w_n\}$.
Then, for any vector v in $\mathbb{R}^n$ (in the subspace or not), we can set:

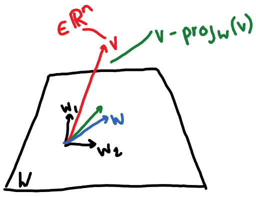
$$proj_w(v) = \frac{w_1\cdot v}{w_1\cdot w_1}\cdot w_1 + \cdots + \frac{w_m\cdot v}{w_m\cdot w_m}\cdot w_m$$

This is the orthogonal projection of v onto w.

Theorem

Let W be a subspace of $\mathbb{R}^n$ and $v\in\mathbb{R}^n$.
Then,

1) $proj_w(v) \in$ W
2) $v - proj_w(v)$ is orthogonal
3) $proj_w(v)$ is the best approximation of v by vectors in W
4) $proj_w(v)$ is the only vector that satisfies 1) and 2)



Example

$$w_1 = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, w_2 = \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}, w_3 = \begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$$

$$proj_u\begin{pmatrix}1\\-1\\2\\-1\end{pmatrix} = 0\cdot\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \frac{-2}{2}\cdot\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} + \frac{3}{2}\cdot\begin{pmatrix}0\\0\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\-1\\\frac{3}{2}\\\frac{-3}{2}\end{pmatrix}$$

--^test 4 material

19.4 Gram-Schmidt algorithm

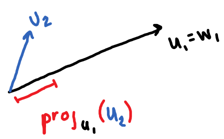
Theorem

Let $\{u_1,\dots u_m\}$ be a basis of a subspace U of $\mathbb{R}^n$.
Then, define:

$w_i := u_1$

$w_2 := u_2 - proj_{v_1}(u_2)$

$\quad = u_2 - \frac{w_1\cdot u_2}{w_1\cdot w_1}\cdot w_1$



$w_3 := u_3 - proj_{v_2}(u_3)$ where $V_2 = span\{u_1, u_2\}$

$= u_3 - \frac{w_1\cdot u_3}{w_1\cdot w_1}\cdot w_1 - \frac{w_2\cdot u_3}{w_2\cdot w_2}\cdot w_2$

$\vdots$

$w_m := u_m - proj_{v_{m-1}}(u_m)$

$\quad = u_m - \frac{w_1\cdot u_m}{w_1\cdot w_1}\cdot w_1 - \cdots - \frac{w_{(m-1)}\cdot u_m}{w_{m-1}\cdot w_{m-1}}\cdot w_{m-1}$

This, $\{w_1,\dots,w_m\}$ is an orthogonal basis for U.